GENERALIZATION OF THE NONLINEAR-RESISTANCE METHOD TO AN ARBITRARY THERMAL-CONDUCTIVITY – TEMPERATURE RELATIONSHIP FOR STEADY-STATE HEAT-CONDUCTION PROBLEMS

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The method previously proposed by the author for solution of nonlinear heat-conduction problems is extended to the case in which there is an arbitrary relationship between the thermal conductivity and the temperature. The possibility of using the method to solve nonsteady-state problems is considered.

The development of a method for solving nonlinear heat-conduction problems is a current problem of great interest; among other approaches, analog-computer techniques have been enlisted [1].

Nonlinear problems have been solved with the aid of structural models [2], static integrators [3], continuous-media models [4], as well as resistance networks [5].

Without analyzing the analog methods in detail, we note that they are often very cumbersome, requiring the use of successive approximations and readjustment of model components.

A new method was proposed in [6] for simulation of nonlinear steady-state heat-conduction problems; it was later called the method of nonlinear resistances. With this approach, a Schneider substitution [7] is used to reduce the nonlinear equation for steady-state heat conduction to a Laplace equation, while the boundary conditions (for the third boundary-value problem) becomes nonlinear. Next, departing from the traditional method of simulating the thermal resistance on the boundary by means of ordinary resistors, the author proceeded in a very logical way to simulate the thermal nonlinearity by means of an electrical nonlinearity, using elementary nonlinear resistors (incandescent bulbs, current regulators, etc.). With this approach, the method is solved in one pass, with no need for successive approximations or linearization of the boundary conditions, as is necessary when other methods are employed. Moreover, it becomes possible to use simple analog devices to solve nonlinear problems: networks of fixed resistors and models constructed from electrically conducting paper.

The method of nonlinear resistances does have a substantial drawback: it involves the use of the Schneider transformation which yields satisfactory results only when there is a linear relationship between the thermal-conductivity coefficient and the temperature. Nonlinearities that occur for a more complex $\lambda = f(t)$ relationship, and requiring other transformations, cannot be simulated by means of nonlinear elements. Although most of the materials employed in the construction of machines have a $\lambda = f(t)$ characteristic that is nearly linear, nonetheless if the method is to have a high degree of universality, it must be extended to the case in which there is an arbitrary relationship between the thermal conductivity and the temperature.

A great help in the effort to solve this problem was the utilization of electron tubes as nonlinear elements, as proposed by the author, together with V. E. Prokof'ev, and a specially constructed device for adjusting the characteristics of the nonlinear resistors, which permitted very rapid and accurate fitting of the tube plate characteristic to the curve reflecting the specified nonlinearity. Investigation of the way in which various parameters affect the tube plate characteristic has shown that by changing the bias voltage

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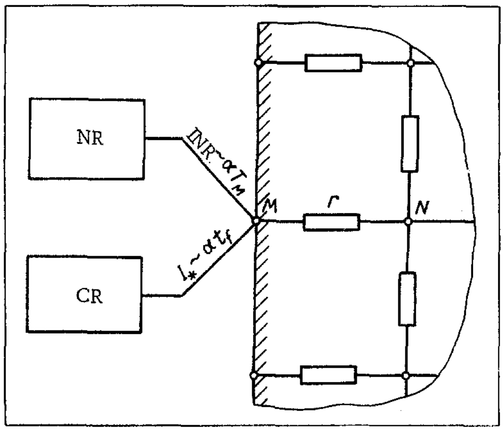

Fig. 1. Realization of nonlinear boundary conditions of the third kind by means of an electric model: NR) nonlinear resistor; CR) current regulator.

on the grids, and by varying the resistance connected in parallel with the tube, it is possible to obtain almost any parabolic plate characteristic, with various exponents and coefficients for the parabola.

Using an example in which the thermal conductivity is a linear function of the temperature, we first show that to solve nonlinear heat-conduction problems by means of the nonlinear-resistance method we can avoid the Schneider substitution by using the Kirchhoff transformation, which is more universal, and is suitable for an arbitrary $\lambda = f(t)$ relationship.

The nonlinear equation of steady-state heat conduction (we consider the two-dimensional case for simplicity)

$$\frac{\partial}{\partial x}\left[\lambda(t)\frac{\partial t}{\partial x}\right] + \frac{\partial}{\partial y}\left[\lambda(t)\frac{\partial t}{\partial y}\right] = 0 \tag{1}$$

can be reduced with the aid of the expression

$$\theta = \int_0^t \lambda(t)\,dt \tag{2}$$

to the Laplace equation

$$\nabla^2\theta = 0, \tag{3}$$

as has been shown previously, while the nonlinear boundary condition of the third kind (the Dirichlet and Neumann problems are of no particular interest, since the further solution is the same as that for linear problems)

$$\alpha(t - t_f) = -\lambda(t)\frac{\partial t}{\partial n} \tag{4}$$

can be transformed, when there is a linear relationship between the thermal conductivity and the temperature,

$$\lambda = a + bt, \tag{5}$$

to the form

$$\alpha\left[-\frac{a}{b} + \left(\frac{a^2}{b^2} + \frac{2}{b}\theta\right)^{0,5} - t_f\right] = -\frac{\partial\theta}{\partial n}. \tag{6}$$

We let

$$t_f + \frac{a}{b} = \theta_*^{0,5} = \text{const},$$
$$-\frac{a^2}{2b} = \theta_0 = \text{const}, \tag{7}$$

and then (6) is written as

$$\alpha\left[(\theta - \theta_0)^{0,5} - \theta_*^{0,5}\right] = -\frac{\partial\theta}{\partial n} \tag{8}$$

or, after going over to finite-difference form,

$$\alpha\left[(\theta_M - \theta_0)^{0,5} - \theta_*^{0,5}\right] = -\frac{1}{h}(\theta_M - \theta_N), \tag{9}$$

where θ_M and θ_N are, respectively, the values of the function θ at the boundary point M and at the nearby interior point N, one step away.

If the boundary condition of the third kind is realized by an electric model, as shown in Fig. 1, we can write the following Kirchhoff law for the boundary point M:

$$I_* - A(V_M - V_0)^n - \frac{1}{r}(V_M - V_N) = 0, \tag{10}$$

where I_* is the current supplied to the boundary point by the current regulator; V_M, V_N are the potentials at M and N; A, n are the coefficient and exponent of the tube plate characteristic; V_0 is the potential applied to the cathode.

It should be remembered that tube plate characteristics form a family of curves of the type

$$I = AU^n. \tag{11}$$

Investigation of tube characteristics shows that such curves are characteristic of many triodes, beam-power tetrodes, pentodes, and heptodes.

We modify (10) so that it takes a form similar to that of (9). To do this, we represent the function I_* as

$$I_* = AV_*^n. \tag{12}$$

After this we rewrite (10) as

$$A\left[(V_M - V_0)^n - V_*^n\right] = -\frac{1}{r}(V_M - V_N). \tag{13}$$

It is easy to see that (9) and (13) will be identical if the exponent n equals 0.5 and if the following condition is satisfied [6]:

$$\frac{Ar\theta_*^{0.5}}{\alpha h V_*^{0.5}} = 1. \tag{14}$$

Thus to realize the boundary condition (9) by means of an electric model we must do the following: specify a particular value of I_*, use the formula

$$A^2 = \frac{\alpha h I_*}{r\theta_*^{0.5}}, \tag{15}$$

which follows from (14) and (12), to find the coefficient A, and select the tube biases and the value of the parallel resistor so that the tube characteristic coincides with the relationship $I = AU^{0.5}$. In this case, the tube cathode is not at zero potential, as must be the case when the Schneider substitution is used, and the relationship $I = AV^{0.5}$ is simulated; instead we use the potential V_0, found from the formula

$$V_0 = m_\theta \theta_0, \tag{16}$$

where m_θ is a conversion factor that permits us to go from θ to V; it is found as

$$m_\theta = \frac{V_*}{\theta_*}. \tag{17}$$

In this expression, θ_* is known from the conditions of the problem, while V_* is determined from (12) on the basis of the specified I_* and the value found for A. Incidentally, this coefficient m_θ is also needed to interpret the results obtained from the model.

Since θ_0 can be either positive or negative in accordance with (7), the potential V_0 must be supplied with the appropriate sign.

Going over to the general case in which there is an arbitrary relationship between the thermal conductivity and the temperature, and making use of the same transformation (2), we represent (4) as

$$\alpha T(\theta) - \alpha t_f = -\frac{\partial\theta}{\partial n}, \tag{18}$$

where $T(\theta)$ is a function that is the inverse of $\theta(t)$, and is determined by inverting (2).

The right side of (18), represented in finite-difference form as was done in (9), is simulated directly by means of the network. The constant term on the right side, representing the product of the heat-transfer coefficient α and the temperature t_f can be simulated, as shown in Fig. 1, by current supplied from a regulator. As for the first (nonlinear) term on the left side of (18), if the function $T(\theta)$ can be represented in finite form as, for example, in the present case, it can be simulated, it is only necessary to adjust the tube characteristic to a curve whose nature depends on the form of the function $T(\theta)$.

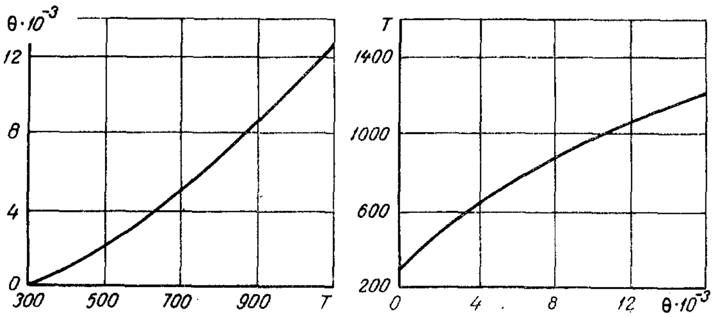

Fig. 2. Inversion of function $\theta = \int_0^t \lambda(t)\,dt$ for type ÉI-612 austenitic steel ($\lambda = 4.42 + 1.94 \cdot 10^{-2}T$).

If the function $\theta = \theta(t)$ is such that it is not easy to obtain the inverse function in finite form, representing $\theta = \theta(t)$ as the series

$$\theta = at + bt^2 + ct^3 + dt^4 + \cdots, \tag{19}$$

we can obtain the inverted form

$$t = T(\theta) = a'\theta + b'\theta^2 + c'\theta^3 + d'\theta^4 + \cdots, \tag{20}$$

where

$$a' = \frac{1}{a}, \quad b' = -\frac{b}{a^3}, \quad c' = \frac{1}{a^5}(2b^2 - ac),$$

$$d' = \frac{1}{a^7}(5abc - a^2d - 5b^3).$$

It may prove still simpler to invert the function graphically, i.e., having the graph of $\theta = \theta(t)$, to construct the relationship $T = T(\theta)$ (Fig. 2). To finish solution of the problem, we simulate the product of the heat-transfer coefficient by the resulting function $T = T(\theta)$. This can be done if the nonlinear electrical resistance has the characteristic

$$I \sim \alpha T(\theta). \tag{21}$$

As we have already noted, almost any relationship between the current through the nonlinear resistor and the voltage across it can be obtained by appropriate variation of the bias voltages on the tube grids and of the resistor connected in parallel with the tube.

Equation (10) can be written in the simpler form

$$I_* - I_{NR} = \frac{1}{r}(V_M - V_N). \tag{22}$$

Here I_{NR} is the current flowing from the boundary point through the nonlinear resistor.

If we introduce the scale factors

$$m_i = \frac{\alpha t}{I}, \quad m_r = \frac{r}{h}, \quad m_V = \frac{\theta}{V}, \tag{23}$$

then the equation

$$\alpha T_M(\theta) - \alpha t_f = -\frac{1}{h}(\theta_M - \theta_N) \tag{24}$$

will be the finite-difference interpretation of (18), and can be rewritten as

$$m_i(I_1 - I_2) = -m_r m_V \frac{1}{r}(V_M - V_N), \tag{25}$$

where

$$m_i I_1 = \alpha T_M(\theta),$$
$$m_i I_2 = \alpha t_f.$$

If the currents I_* and I_{NR} (Fig. 1) are so specified that

$$I_* = I_2, \quad I_{NR} = I_1, \tag{26}$$

then (22) and (25) will be identical provided the following condition is satisfied:

$$\frac{m_r m_V}{m_i} = 1. \tag{27}$$

Now we need only adjust the nonlinear resistance in appropriate fashion.

Adjustment of the nonlinear resistance is simplest if we employ the instrument mentioned above to fit the characteristics of the nonlinear elements. The approach used with this device, whereby the plate characteristic is fitted to a reference parabola on the screen of an oscilloscope, cannot be considered to be universal. This is particularly true when the $T(\theta)$ cannot be represented analytically. In such case, the construction of the reference parabola is a problem in its own right, whose solution is often far from simple.

Thus in solving problems involving a more complicated relationship between thermal conductivity and temperature than a linear dependence, the nonlinear-resistance characteristic should not be fitted to a reference parabola, but directly to the $T = T(\theta)$ relationship, obtained by one of the methods discussed above. The fact is that even the most superficial analysis of (24)-(26) shows that the relationship between the current through the nonlinear resistor and the voltage across it only differs from $T = T(\theta)$ in a constant factor. As a consequence, the plate characteristic of the nonlinear element will only differ by a scale factor from the inverted function $T = T(\theta)$, and provision can be made for this by appropriate choice of the oscilloscope gains.

The criterial relationship (27) and Eqs. (23) are used to determine the parameters of the model. To conclude, we felt that the method discussed can be extended successfully to nonsteady-state nonlinear heat-conduction problems. The sole difference lies in the fact that during solution of the problem, the characteristics of the nonlinear element change in time. This is done by varying the bias on the tube grid in accordance with a predetermined law, which is realized by means of channels available in the function-generator and boundary-condition elements available on modern analog computers.

NOTATION

t is the temperature of the body;
t_f is the ambient temperature;
λ is the thermal-conductivity coefficient;
α is the heat-transfer coefficient;
h is the net spacing;
I is the current;
V is the potential;
U is the potential difference;
r is the resistance corresponding to the net spacing;
m is a scale factor;
A is a coefficient of proportionality;
n is the exponent;
NR is the nonlinear resistor;
CR is the current regulator.

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